

# A shear spectral sum rule in a non-conformal gravity dual

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A sum rule which relates a stress-energy tensor correlator to thermodynamic functions is examined within the context of a simple non-conformal gravity dual. Such a sum rule was previously derived using AdS/CFT for conformal  $\mathcal{N} = 4$  Supersymmetric Yang-Mills theory, but we show that it does not generalize to the non-conformal theory under consideration. We provide a generalized sum rule and numerically verify its validity. A useful byproduct of the calculation is the computation of the spectral density in a strongly coupled non-conformal theory. Qualitative features of the spectral densities and implications for lattice measurements of transport coefficients are discussed.

## I. INTRODUCTION

Sum rules are powerful tools useful in the exploration of nonperturbative phenomena. The use of sum rules in this way dates back nearly three decades now [1]; but recently there has been some interest in applying these tools to the strongly coupled plasma created at the Relativistic Heavy Ion Collider (RHIC) [2–5]. In [6, 7], the authors used the low energy theorems of [8] to write down a sum rule which relates an integral over the spectral density to thermodynamic quantities. In general such a sum rule provides constraints on the spectral function, which could itself be used to extract transport coefficients (via Kubo’s formulas). Constraints or some knowledge of the functional form of the spectral density are generally needed in order to have any hope of extracting transport coefficients from the lattice. The authors of [6, 7] were able to argue for qualitative features of the bulk viscosity of the quark-gluon plasma (QGP) by combining their sum rule with both an ansatz for the spectral density and lattice data. This approach was later criticized in [9], and later some corrections and clarifications were added in [10]. This latter work also derived several other sum rules using Kramers-Kronig relations. Subsequent works have derived additional sum rules, and examined the applications of such sum rules to lattice computations [11–13]. One such sum rule derived in [10] was derived using AdS/CFT and was found to be applicable to  $\mathcal{N} = 4$  supersymmetric Yang-Mills (SYM) theory. It is the aim of this paper to examine this sum rule within the context of a non-conformal gravity dual theory using the tools of the AdS/CFT correspondence [14–17].

The sum rule in question is [10]

$$\frac{2}{5}\varepsilon = \frac{2}{\pi} \int_0^\infty \frac{dw}{w} [\rho^{\text{shear}}(w) - \rho_{T=0}^{\text{shear}}(w)], \quad (1)$$

Here  $\varepsilon$  is the energy density, and  $\rho$  is the spectral density

$$\rho^{\text{shear}}(w) \equiv -\text{Im} G_R^{\text{shear}}(w). \quad (2)$$

The retarded Green’s function in the “tensor” channel is defined as:

$$G_R^{\text{shear}}(w) \equiv -i \int d^4x e^{iwt} \langle [T^{xy}(x), T^{xy}(0)] \rangle \theta(t) \quad (3)$$

and the subscript  $T = 0$  means the quantity of interest is evaluated in the limit of zero temperature.

Both sides of the sum rule can be computed using AdS/CFT techniques. The left side depends only on thermodynamic quantities, which are easily evaluable for the theory of interest. In order to evaluate the right hand side, one needs to compute the spectral density  $\rho$  as a function of  $w$ . In AdS/CFT, the differential equations necessary to compute spectral densities are often difficult to solve analytically (though in some cases analytical results have been given in the literature [18]). In this work we will solve the differential equations numerically, and hence our verification of the sum rule will be numerical in nature.

In [10], the authors checked that the left and right sides of the sum rule (1) are in agreement within the context of the (conformal)  $\mathcal{N} = 4$  SYM theory. However, the authors then state that the sum rule should hold for *any* Einstein gravity dual. As shown below, this is actually not the case. We have evaluated the left and right sides of the sum rule (1) in a particular non-conformal gravity dual theory and find that the left side is not, in general, equal to the right side. In fact, one should expect that the sum rule should be corrected as

$$\frac{2}{5}\varepsilon + F(\varepsilon, P, v_s) = \frac{2}{\pi} \int_0^\infty \frac{dw}{w} \Delta \rho^{\text{shear}}(w). \quad (4)$$

We will often employ the shorthand notation  $\Delta$  to denote a quantity which has the zero temperature part subtracted out. For example,

$$\Delta \rho(w) \equiv \rho(w) - \rho_{T=0}(w). \quad (5)$$

To be consistent with currently known results,  $F(\varepsilon, P, v_s)$  must vanish when  $\varepsilon = 3P$  and  $v_s^2 = 1/3$ .

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Using the same techniques as in [10], we have been able to derive the correction of the left hand side of the sum rule in our particular non-conformal gravity dual. We explicitly show that the left and right sides of our corrected sum rule (4) agree within the numerical error.

The non-conformal theory in which we work is a simple 5D single scalar model with an exponential potential. This model is sometimes called the Chamblin-Reall model [19], and has been extensively studied in the literature [20–26]. We emphasize that this model is not particularly well suited for QGP phenomenology; it has no conserved charge, and also has the peculiar feature of being both non-conformal and having a speed of sound which is independent of temperature. Still, we choose to work in this model because it is perhaps the simplest example of a non-conformal gravity dual where many of the hydrodynamic equations can be solved exactly. It is worth mentioning that *if* there is a precise field theory dual to this model, it is not known at present. However, recently it was found that the dynamics of a more complicated string theory setup (including fundamental flavors) were captured by an effective single scalar Chamblin-Reall background [27]. This may indicate a connection between the Chamblin-Reall background and more rigorous non-conformal deformations of  $\mathcal{N} = 4$  SYM theory.

Our paper is organized as follows. In Sec. II, we introduce our non-conformal gravitational dual, the single scalar Chamblin-Reall background. In Sec. III, we present the details of the evaluation of the right hand side of the sum rule. This section involves introducing a tensor perturbation into the geometry, numerically solving for the spectral density, and integrating the result. In Sec. IV, we evaluate the left side of the sum rule using the known thermodynamics of the gravity background; it is evident that the left side does not agree with the right side except in the limiting case of a conformal theory. We then proceed to derive the correct form of the left side and present an improved sum rule where the left and right sides agree numerically. We conclude the paper in Sec. V. In Appendix A we discuss the relevant sum rule within the context of (weakly coupled) Yang-Mills theory. Other technical details of our calculations and useful reference formulae are found in Appendices B - D.

## II. GRAVITY BACKGROUND

The theory under consideration is a 5D gravitational dual generated by a single scalar field<sup>1</sup>

$$\mathcal{S} = \frac{1}{2\kappa} \int d^5x \sqrt{-g} \left[ R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] + \frac{1}{\kappa} \int d^4x \sqrt{-\gamma} \theta, \quad (6)$$

<sup>1</sup> Throughout this work, we use the “mostly plus” metric signature.

where  $\kappa$  is related to the five dimensional Newton’s constant,  $\kappa \equiv 8\pi G_5$ . The second term is a boundary contribution, the well known Gibbons-Hawking term which is necessary for a well defined variational principle. The induced metric on the boundary is denoted by  $\gamma_{\mu\nu}$ ,  $\nabla_\mu$  denotes the covariant derivative, and  $\theta$  is the trace of the second fundamental form

$$\theta_{\mu\nu} = \nabla_\mu \hat{n}_\nu \quad (7)$$

with  $\hat{n}^\nu$  a unit vector normal to the boundary.

We will assume the metric is of the “black brane” type

$$ds^2 = g_{tt}(z)dt^2 + g_{xx}(z)d\mathbf{x}^2 + g_{zz}(z)dz^2, \quad (8)$$

and that the coordinates can be chosen such that there is a black brane horizon at  $z = z_h$ . We will often employ the symbol

$$f(z) = -g_{tt}(z)g^{xx}(z). \quad (9)$$

As mentioned in the introduction, in presenting our main results we will specify to a particular type of metric, the Chamblin-Reall background. However, whenever possible, we will keep the metric components general in hopes that doing so may be useful for those wishing to do analogous calculations in different backgrounds.

The Chamblin-Reall background can be found by assuming an exponential potential of the form

$$V(\phi) = -\frac{6}{L^2} \frac{(2-\delta)}{(1-2\delta)^2} \exp \left\{ \sqrt{\frac{4\delta}{3}} \phi \right\}. \quad (10)$$

The potential contains a parameter  $\delta$  related to conformal symmetry breaking; the precise form of the potential above is chosen for future convenience. The resulting metric and scalar field profile which solve Einstein’s equations are

$$ds^2 = b^2(z) \left[ -f(z)dt^2 + d\mathbf{x}^2 + \frac{dz^2}{f(z)} \right] \quad (11)$$

$$b(z) = \left( \frac{L}{z} \right)^{\frac{1}{1-2\delta}} \quad (12)$$

$$f(z) = 1 - \left( \frac{z}{z_h} \right)^{\frac{2(2-\delta)}{1-2\delta}} \quad (13)$$

$$\phi(z) = -\sqrt{12\delta} \log[b(z)]. \quad (14)$$

Here  $L$  is a constant which is related to the radius of curvature of the space, and  $z_h$  is the position of the horizon. The coordinate  $z$  runs from 0 to  $z_h$ , with the UV boundary at  $z = 0$ .

Thermodynamics and transport coefficients have been studied in this setup in [20, 21, 25, 26, 28]. The relevant results for our purposes are

$$\varepsilon = \frac{3}{1-2\delta} P = \frac{3}{2(2-\delta)} Ts \quad (15)$$

$$v_s^2 = \frac{1}{3}(1-2\delta) \quad (16)$$

$$\frac{\zeta}{\eta} = 2 \left( \frac{1}{3} - v_s^2 \right) \quad (17)$$

Here we have introduced  $s$  as the entropy density,  $P$  as the pressure, and  $v_s$  as the speed of sound. Note that the parameter  $\delta$  is a measure of the conformal symmetry breaking; for  $\delta = 0$ , we recover the usual  $AdS_5$  metric, which is dual to a conformal field theory. One should also note that this setup is rather peculiar in that the speed of sound is *constant* with respect to temperature, though it is not necessarily equal to  $1/\sqrt{3}$ . We will always work in the regime where  $0 \leq \delta < 1/2$ ; in this regime the speed of sound is positive and less than  $1/\sqrt{3}$ .

### III. RIGHT SIDE OF SUM RULE

#### A. Tensor mode perturbations

In order to access the two point correlation functions, we must add perturbations to this geometry. We assume a perturbation which depends only on time and the extra-dimensional coordinate  $z$ .

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu}(t, z) \quad (18)$$

$$\phi \rightarrow \phi_0 + \delta\phi(t, z). \quad (19)$$

In general, we might also assume a spatial dependence for the perturbations. Upon Fourier transform, we would acquire a momentum dependence of  $e^{i\mathbf{k}\cdot\mathbf{x}}$ . The sum rule in question involves the two point correlation functions at vanishing spatial momentum so we have set  $\mathbf{k}$  to zero; this is equivalent to assuming the perturbation does not depend on the spatial coordinates  $x^i$ .

Perturbations in the 4D fluid are generally categorized into scalar, vector, and tensor modes denoting their transformation properties under spatial rotations. We will examine the tensor mode; this is the mode which gives access to the shear viscosity. In this case, the only nonzero metric perturbation is  $h_{xy}$ , and we need not consider the fluctuation  $\delta\phi$ , as it does not couple to the metric perturbation in this channel.

In order to compute the correlation functions one must solve the linearized Einstein equations for the perturbation's profile. Once this is accomplished, one must plug the result back into the action and use the prescription of Son and Starinets [17] to get the correlation functions.

The linearized Einstein equation of motion for the Fourier transform

$$H(t, z) \equiv h_y^x(t, z) = \int \frac{dw}{2\pi} H(w, z) e^{-iwt} \quad (20)$$

is,

$$\frac{1}{\sqrt{-g}g^{zz}} \partial_z [\sqrt{-g}g^{zz} H'] - w^2 g_{zz} g^{tt} H = 0. \quad (21)$$

Throughout this work, we use the prime to denote derivative with respect to the coordinate which labels the extra dimension (in the case at hand,  $z$ ). This equation needs to be solved with the “incoming wave” boundary condition which can be applied by making the ansatz

$$H(z) = f(z)^{-i\omega/2} Y(z), \quad (22)$$

and requiring that  $Y$  is a regular function of  $z$  at the horizon. We have defined the customary dimensionless frequency

$$\mathfrak{w} \equiv w/(2\pi T). \quad (23)$$

The solution for  $H$  will contain one integration constant, which can be related to the boundary value of  $H(z \rightarrow 0)$ .

Correlation functions of the operator dual to the fluctuation  $H$  can be found from the on-shell action. In order to access two point functions, one needs to expand the gravitational action to second order in perturbation  $H$ . Upon application of the equations of motion (“on-shell”), the action reduces to boundary terms, some of which arise due to integration by parts. We will not go through the details of this procedure, but they can be found in complete generality in [29–31]. Here, we present the on-shell boundary terms for the theory in question: a gravity dual with a black brane metric supported by a single scalar field.

In writing the following expressions, we have chosen to remove all instances of the potential with the background equations of motion. These equations can be found in Appendix B. We have also made use of the fact that all backgrounds generated by scalar fields only must satisfy [21]

$$g_{zz}(z) = c_1 [g_{xx}(z)]^4 \frac{[f'(z)]^2}{f(z)}. \quad (24)$$

This constraint is a consequence of the background Einstein equations. Here,  $c_1$  is a constant related to the temperature

$$c_1 = \frac{1}{(4\pi T)^2 g_{xx}(z_h)^3}. \quad (25)$$

Combining these expressions with the Beckenstein-Hawking entropy law  $s = g_{xx}(z_h)^{3/2}/4G$  allows one to remove Newton's gravitational constant  $G$  in favor of the thermodynamic quantities  $T, s$ , and the constant  $c_1$ . In computing the on-shell action, we find the constant  $c_1$  drops out completely. The on-shell action is divergent and so we introduce  $\epsilon$  as a UV cutoff<sup>2</sup>. There are two contributions to the on-shell action which can be written in terms of the quantity

<sup>2</sup> One should take care to distinguish the UV cutoff  $\epsilon$  from the

energy density  $\varepsilon$ .

$$\mathcal{S}[A(z), B(z)] \equiv \frac{TsV}{2} \int_{z=\epsilon} \frac{dw}{2\pi} \frac{f(z)}{f'(z)} \left\{ A(z) H'(w, z) H(-w, z) + B(z) H(w, z) H(-w, z) \right\}. \quad (26)$$

We have introduced  $V$  to indicate the spatial volume, the result of the integration over  $d^3x$ . One contribution to the on-shell action is the results from the fact that bulk action (the first term in (6)) reduces to a total derivative upon application of the equations of motion:

$$\mathcal{S}_{\text{bulk}} = \mathcal{S}[-3, -\mathcal{D}_L[g_{xx}(z)]] \quad (27)$$

We often employ the notation  $\mathcal{D}_L$  to denote the logarithmic derivative

$$\mathcal{D}_L[X] = X'/X. \quad (28)$$

In addition, there is a contribution from the Gibbons-Hawking term (the second term in (6)), which is

$$\mathcal{S}_{\text{GH}} = \mathcal{S}[4, \mathcal{D}_L[f(z)g_{xx}^4(z)]] \quad (29)$$

In total, then

$$\mathcal{S}_{\text{Total}} = \mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{GH}} = \mathcal{S}[1, \mathcal{D}_L[f(z)g_{xx}^3(z)]] \quad (30)$$

In general, the on-shell action needs to be regularized with the addition of counter terms. However, if one is only interested in the imaginary part of the correlators (the spectral density  $\rho$ ), this is not necessary. It is well known by now that the imaginary part of the on-shell action independent of the radial coordinate and is not divergent. The imaginary part is<sup>3</sup>

$$\text{Im } \mathcal{S}_{\text{Total}} = \frac{1}{2i} [\mathcal{S}_{\text{Total}}(w) - \mathcal{S}_{\text{Total}}(-w)] \quad (31)$$

which can be written

$$\text{Im } \mathcal{S}_{\text{Total}} = \frac{TsV}{4i} \int \frac{dw}{2\pi} \frac{f}{f'} \left[ H'(w, z) H(-w, z) - H'(-w, z) H(w, z) \right] \quad (32)$$

As mentioned above, one can evaluate this quantity at any value of  $z$ ; a convenient one is  $z = z_h$ . We assume that our function  $f$  vanishes linearly at the horizon;

$$f(z = z_h) = f_0(z - z_h) + \mathcal{O}((z - z_h)^2), \quad (33)$$

with  $f_0$  a constant. Furthermore, note that the incoming wave boundary conditions require

$$H'(w, z_h)(z - z_h) = -\frac{i\mathfrak{w}}{2} H(w, z_h) [1 + \mathcal{O}(z - z_h)] \quad (34)$$

very near the horizon. Thus,

$$\text{Im } \mathcal{S}_{\text{Total}} = -\frac{VTs}{4} \int \frac{dw}{2\pi} \mathfrak{w} \times H(-w, \epsilon) \left[ \frac{H(w, z_h) H(-w, z_h)}{H(-w, \epsilon) H(w, \epsilon)} \right] H(w, \epsilon). \quad (35)$$

Here,  $H(w, \epsilon)$  is the boundary value of the perturbation as  $z \rightarrow 0$ . The prescription of Son and Starinets states that the two point correlation function of the operator dual to  $H$  is given by [17]

$$-\text{Im } G_R^{\text{shear}} = \frac{Ts\mathfrak{w}}{2} \left[ \frac{H(w, z_h) H(-w, z_h)}{H(-w, \epsilon) H(w, \epsilon)} \right] \quad (36)$$

or, equivalently,

$$\rho(w) = \frac{sw}{4\pi} \left[ \frac{Y(w, z_h) Y(-w, z_h)}{Y(-w, \epsilon) Y(w, \epsilon)} \right] \quad (37)$$

In the case of  $w = 0$ , the only solution to the equations of motion which obeys the boundary conditions is  $H = \text{constant}$ , and thus,

$$\eta = \lim_{w \rightarrow 0} \frac{\rho(w)}{w} = \frac{s}{4\pi} \quad (38)$$

The first equality is the Kubo formula for the shear viscosity. This is now a familiar result.

## B. Numerical computation of spectral density

We are interested in the quantity  $\rho$  at finite values of  $w$ , and thus the equations of motion must be solved numerically. The first step towards this end is to pass to a more convenient coordinate system. We define a dimensionless coordinate  $u$  such that the horizon is at  $u = 1$  and the boundary is at  $u = 0$ . One can write the metric as:

$$ds^2 = \frac{u^{\frac{2}{\delta-2}}}{\alpha} [-dt^2 f(u) + d\mathbf{x}^2] + \frac{u^{\frac{2(2+\delta)}{\delta-2}}}{\alpha(2\pi T)^2} \frac{du^2}{f(u)} \quad (39)$$

$$f(u) = 1 - u^2 \quad (40)$$

The constant  $\alpha$  is not important for our purposes (it drops out of the equations of motion), but for completeness, it is

$$\alpha = \left[ \frac{2(2-\delta)}{4\pi T L(1-2\delta)} \right]^{\frac{2}{1-2\delta}} \quad (41)$$

In this coordinate system, the equation of motion (21) becomes

$$H''(u) - \frac{1+u^2}{u(1-u^2)} H'(u) + \mathfrak{w}^2 \frac{u^{\frac{2(1+\delta)}{\delta-2}}}{(1-u^2)^2} H(u) = 0. \quad (42)$$

<sup>3</sup> This is because the on-shell action  $\mathcal{S}$  is related to the retarded Green's function  $G_R$ , which satisfies  $G_R(w)^* = G_R(-w)$ .

Upon insertion of the incoming wave ansatz (22), the equation for  $Y$  is

$$Y'' - \frac{1+u^2(1-2i\mathfrak{w})}{u(1-u^2)}Y' - \frac{\left(1-u^{\frac{6}{5-2}}\right)u^2\mathfrak{w}^2}{(1-u^2)^2}Y = 0. \quad (43)$$

We use Mathematica's NDSolve function [32] to solve this equation numerically for a given value of  $\mathfrak{w}$  and  $\delta$ . Boundary conditions must be specified. The function  $Y$  must be regular at the horizon in order to comply with the incoming wave boundary condition; the easiest way to apply this condition is to begin integration at some value of  $u$  close to the horizon and specify

$$Y(1-\epsilon) = 1. \quad (44)$$

One also needs to specify the derivative  $Y'(u)$  here. Expanding the equation (43) in powers of  $(1-u)$ , one finds that the leading order term leads to the condition

$$Y'(1-\epsilon) = \frac{3i\mathfrak{w}^2}{2(i+\mathfrak{w})(2-\delta)}Y(1-\epsilon). \quad (45)$$

To summarize, our numerical method is as follows

1. Specify a value of  $\delta$  and  $\mathfrak{w}$ .
2. For these values of  $\delta, \mathfrak{w}$ , use Mathematica's NDSolve to numerically integrate (43) with the boundary conditions (44),(45). Start the integration near the horizon at  $u = 1 - \epsilon$ , and integrate down very near the boundary at  $u = \epsilon$ .
3. Using the now known values of  $Y(\epsilon)$  and (37) one can determine the spectral density<sup>4</sup>.

$$\frac{\rho^{\text{shear}}(\mathfrak{w})}{\mathfrak{w}} = \frac{Ts}{2} \frac{1}{|Y(\epsilon)|^2}. \quad (46)$$

### C. Zero temperature subtraction

The quantity that enters the sum rule is the zero temperature subtracted spectral density. One can compute  $\rho^{\text{shear}}(w)_{T=0}$  analytically. A gravitational metric dual to a zero temperature field theory is, intuitively, one without a black brane horizon. Returning now to our original  $z$  coordinates, we set  $f(z) = 1$ . In terms of  $b(z)^2 = g_{xx}(z)$ , the equations of motion become

$$H_0''(z) + \mathcal{D}_L[g_{xx}^{3/2}(z)]H_0'(z) + w^2H_0(z) = 0. \quad (47)$$

We are using  $H_0(z)$  to denote the solution at zero temperature. We are interested in the case where  $g_{xx}(z) = \left(\frac{L}{z}\right)^n$ , with  $n$  being a function of  $\delta$ .

$$n \equiv \frac{2}{1-2\delta}. \quad (48)$$

The equation is then

$$H_0''(z) - \frac{3n}{2z}H_0'(z) + w^2H_0(z) = 0. \quad (49)$$

This equation is solved in terms of Bessel functions, or alternatively in terms of Hankel functions of the first and second kind:

$$H_0(z) = z^{\frac{2+3n}{4}} \left[ C_1 h_{\frac{2+3n}{4}}^{(1)}(wz) + C_2 h_{\frac{2+3n}{4}}^{(2)}(wz) \right]. \quad (50)$$

The combination  $(2+3n)/4$  appears frequently in our calculations, and for simplicity we will use the definition

$$l \equiv \frac{2+3n}{4} = \frac{2-\delta}{1-2\delta}. \quad (51)$$

The Hankel functions of the first kind behave at  $z \rightarrow \infty$  as  $\sim e^{iwz}/\sqrt{z}$ , whereas the Hankel functions of the second kind behave as  $\sim e^{-iwz}/\sqrt{z}$ . One can think of the zero temperature metric as possessing a ‘‘horizon’’ at  $z = \infty$ , hence we should choose  $C_2 = 0$  so that waves are only traveling towards the ‘‘horizon’’ [17].

To get the correlation functions, we again need to expand the zero temperature on-shell action to quadratic order in the perturbation  $H_0$ . The steps are analogous to those above, only now we are working in a coordinate system where  $-g_{tt} = g_{xx} = g_{zz}$ . The results can be written in terms of the quantity

$$\mathcal{S}^{T=0}[A(z), B(z)] = \frac{sV}{8\pi} \int_{z=\epsilon} \frac{dw}{2\pi} \left( \frac{g_{xx}(z)}{g_{xx}(z_h)} \right)^{3/2} [A(z)H_0'(w, z)H_0(-w, z) + B(z)H_0(w, z)H_0(-w, z)]. \quad (52)$$

<sup>4</sup> One may be worried about applying our results to this new coordinate system; in fact, the only coordinate dependent assumption

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The results for the bulk and Gibbons-Hawking terms are

$$\mathcal{S}_{\text{bulk}}^{\text{T}=0} = \mathcal{S}^{\text{T}=0} [3, \mathcal{D}_L[g_{xx}(z)]], \quad (53)$$

$$\mathcal{S}_{\text{GH}}^{\text{T}=0} = \mathcal{S}^{\text{T}=0} [-4, -\mathcal{D}_L[g_{xx}^4(z)]], \quad (54)$$

$$\mathcal{S}_{\text{Total}}^{\text{T}=0} = -\mathcal{S}^{\text{T}=0} [1, \mathcal{D}_L[g_{xx}^3(z)]]. \quad (55)$$

And again applying the prescription of Son and Starinets, we find the spectral density,

$$\rho_{\text{T}=0}^{\text{shear}}(w) = \frac{s}{8\pi i} \left( \frac{g_{xx}(z_*)}{g_{xx}(z_h)} \right)^{3/2} \times \left[ \frac{H'_0(w, z_*)H_0(-w, z_*) - H'_0(-w, z_*)H_0(w, z_*)}{H_0(w, \epsilon)H_0(-w, \epsilon)} \right], \quad (56)$$

The symbol  $z_*$  is used to denote any particular value of  $z$  which we choose (again, this result is independent of  $z$ ). In computing the finite temperature spectral density, we found it most convenient to evaluate the result at the horizon; here it is more convenient to choose  $z_* = \epsilon$ . Let us also now specify to the case at hand with  $g_{xx} = (L/z)^n$ .

$$\rho_{\text{T}=0}^{\text{shear}}(w) = \frac{s}{8\pi i} \left( \frac{z_h}{\epsilon} \right)^{3n/2} \times \left[ \frac{H'_0(w, \epsilon)H_0(-w, \epsilon) - H'_0(-w, \epsilon)H_0(w, \epsilon)}{H_0(w, \epsilon)H_0(-w, \epsilon)} \right], \quad (57)$$

With the use of the solution (50), one finds

$$\begin{aligned} H'_0(w, \epsilon)H_0(-w, \epsilon) - H'_0(-w, \epsilon)H_0(w, \epsilon) \\ = -\frac{4|C_1|^2(-1)^{-3n/4}(\epsilon)^{3n/2}}{\pi}. \end{aligned} \quad (58)$$

Finally, one needs to employ the expansion

$$h_l^{(1)}(x) = -\frac{i}{\pi} \left( \frac{2}{x} \right)^l \Gamma(l) + \mathcal{O}(x^2) + \mathcal{O}(x^l). \quad (59)$$

Recall that  $l > 2$  in the physical region. Putting it all together we find

$$\rho_{\text{T}=0}^{\text{shear}}(w) = \frac{sw(wz_h)^{3n/2}}{2^{2l+1}\Gamma(l)^2}. \quad (60)$$

Finally, one should remove  $z_h$  in favor of  $T$ . When the metric is written in the coordinate system (11), the Hawking temperature is

$$T = -\frac{f'(z_h)}{4\pi} = \frac{l}{2\pi z_h}. \quad (61)$$

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we have made is that the  $g_{tt}$  component vanish linearly near the horizon, and the  $g_{zz}$  or  $g_{uu}$  component diverge as  $1/x$  near the horizon. These facts are true in both coordinate systems.

Finally, using the definitions for  $n$  and  $l$  (48) and (51), we have our final result for the zero temperature case:

$$\frac{\rho_{\text{T}=0}^{\text{shear}}(\mathfrak{w})}{\mathfrak{w}} = \frac{\pi T s \left[ \frac{\mathfrak{w}}{2} \left( \frac{2-\delta}{1-2\delta} \right) \right]^{\frac{3}{1-2\delta}}}{2\Gamma \left( \frac{2-\delta}{1-2\delta} \right)^2}. \quad (62)$$

After numerically computing the spectral density at finite temperature, one subtracts off this piece to remove the UV divergences at large  $\mathfrak{w}$ .

One drawback of our method is that it requires a large degree of numerical precision; both the zero temperature and finite temperature spectral functions diverge at large  $\mathfrak{w}$ , but their difference approaches zero. One needs to compute both spectral functions to a high degree of precision before performing the subtraction to get the desired result. In the future it would be desirable to build this subtraction into the numerics, to avoid this need for very high precision numerics.

#### D. Numerical results for spectral density

We have numerically computed the (zero-temperature subtracted) shear spectral function for  $\delta = 0, 0.1, 0.2$  and  $0.3$ . Some sample results are shown in Fig. 1. We have computed the spectral functions out to a large value of  $\mathfrak{w}$ ; we cease our numerical computation when the oscillations have reached 0.1% of their maximum value. The qualitative behavior of this spectral function is quite interesting. The spectral density oscillates around the zero temperature result, with the oscillations eventually dying out as one moves to higher frequencies. This oscillation phenomenon has been noticed before in previous computations of the spectral density in  $\mathcal{N} = 4$  SYM theory [33, 34]. These damped oscillations are thought to be a reflection of the pole structure of the retarded correlation functions in the complex plane [18, 35].

When moving to a non-conformal theory, we notice a qualitatively new behavior. As one increases the non-conformal parameter  $\delta$ , the oscillations become initially larger and larger, and the spectral function takes a longer time to settle down to its zero temperature value. This is especially evident for the case of  $\delta = 0.3$  where the oscillations are so large that it does not fit nicely on a plot with those shown in Fig. 1. We are unsure of the physical interpretation of this behavior. However, we can use our spectral functions to compute Euclidean correlation functions, quantities that are computed on the lattice (see for example [36, 37]). The relation between the spectral function and the Euclidean correlators is

$$G_E(\tau) = \frac{1}{\pi} \int dw \rho(w) \frac{\cosh[w(\tau - \beta/2)]}{\sinh[w\beta/2]}, \quad (63)$$

where  $\tau$  is the Euclidean time variable, which has period  $\beta \equiv 1/T$ . Using our numerical results for the spectral density, we plot the results for the Euclidean correlation functions in Fig. 3. It is interesting to note that the

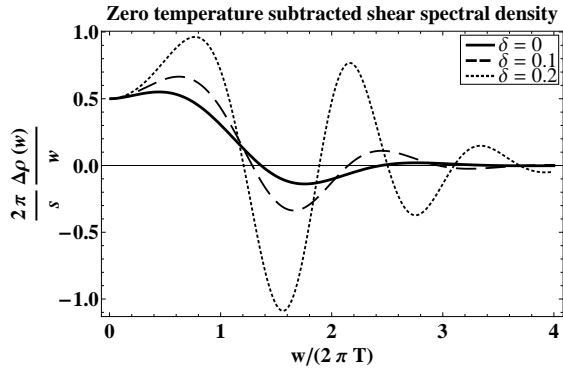


FIG. 1. Plots of the zero-temperature subtracted spectral density versus frequency for several values of  $\delta$ . For large  $w$ , the spectral density always approaches the zero temperature result. For small  $w$ , the slope of the spectral density always approaches the same result which is confirmation that the shear viscosity takes on the universal value  $\eta/s = 1/4\pi$ .

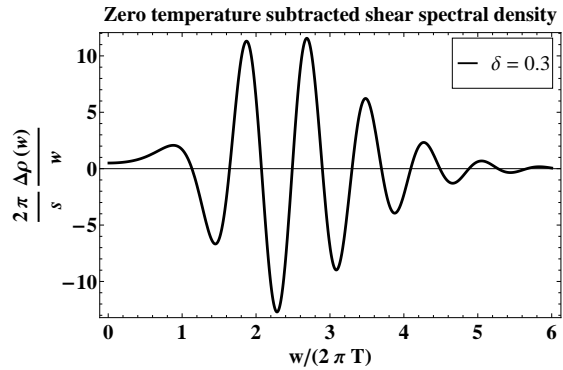


FIG. 2. Plot of the zero-temperature subtracted spectral density versus frequency for  $\delta = 0.3$ . One should note the qualitative differences in the plots as one increases  $\delta$ . For larger values of  $\delta$  the oscillations grow larger and more frequent, and take a longer time to die off. Note the difference in the axis scaling between this plot and Fig. 1.

value of  $\delta$  (which is proportional to the bulk viscosity  $\zeta$ ) affects the magnitude of the Euclidean correlation function quite strongly. This may have implications for lattice measurements, since the Euclidean correlation function is directly measured there. The gravity dual in which we are working is only toy model of a non-conformal theory, and hence we will not make any attempt to extract a value of the bulk viscosity from the lattice data. However, *if* the qualitative behavior noted here persists in more realistic holographic models of physical gauge theories, it suggests that one could possibly gain insight into the value of the bulk viscosity from the *tensor* correlation function considered here. This could prove to be quite useful, since it would provide an independent measurement of the bulk viscosity which is usually extracted via the Kubo relations in the *bulk* channel.

To evaluate the sum rule, one must integrate the spectral functions. Technically, we use Mathematica to per-

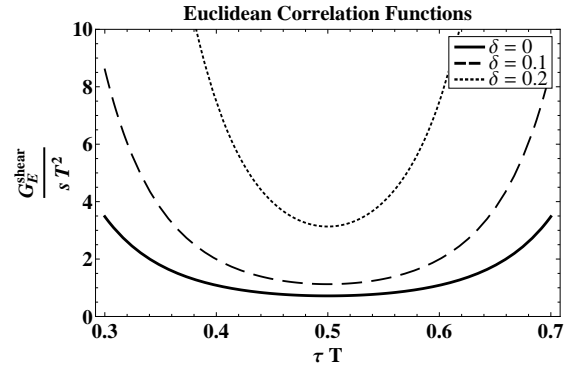


FIG. 3. Plot of the Euclidean correlation function associated with the shear spectral density as a function of the Euclidean time  $\tau$  for various values of  $\delta$ . The value of  $\delta$  (and hence the value of the bulk viscosity in our model) has a strong effect on the shape and magnitude of these functions.

$\delta$	(LHS of Sum rule) $\times \frac{1}{Ts}$	(RHS of Sum rule) $\times \frac{1}{Ts}$
0.0	$\frac{3}{10} \approx 0.300\dots$	0.300...
0.1	$\frac{6}{19} \approx 0.318\dots$	0.326...
0.2	$\frac{1}{3} \approx 0.333\dots$	0.357...
0.3	$\frac{6}{17} \approx 0.353\dots$	0.395...

TABLE I. Results for the left and right sides of the sum rule given in [10]. The left side values come from (1) and (15). The right side values come from numerically integrating our results for the spectral density.

form a cubic interpolation between the points which are spaced at intervals of  $\tau = 0.01$ , and numerically integrate the resulting function. The results are given in Table I.

#### IV. LEFT SIDE OF SUM RULE

If we take the sum rule (1) at face value, we can easily evaluate the left hand side using (15). The results are shown in Table I. Clearly, there is substantial disagreement between the left and right sides. Notice that in the case of  $\delta = 0$ , the two sides are in agreement, which was also the conclusion of [10]. However once we deviate from conformality, differences appear. The error for  $\delta = 0.3$  is greater than 10% which is not accounted for by numerical error.

Thus, we come to the conclusion that the left hand side of the sum rule must be modified. Following [10], the sum rule is, more generally

$$\Delta G_R^{\text{shear}}(w = i\infty) - \Delta G_R^{\text{shear}}(w = 0) = \frac{2}{\pi} \int_0^\infty \frac{dw}{w} \Delta \rho^{\text{shear}}(w). \quad (64)$$

The left side of (64) can be computed using AdS/CFT techniques. The general method we use is the same as

[10], except we are now working in a more general background, and hence the equations of motion and action are modified.

We will first compute the term  $\Delta G_R^{\text{shear}}(w = i\infty)$ . Let us define

$$Q \equiv iw. \quad (65)$$

In terms of the dimensionless variable  $\mathbf{q} = i\mathbf{w}$ , the limit is  $\mathbf{q} \rightarrow \infty$ . Note that this limit can also be achieved by taking  $T \rightarrow 0$ , or equivalently  $z_h \rightarrow \infty$ . This, in turn is equivalent to examining the metric in the regime near the boundary  $z \rightarrow 0$ . Our first step is to transform the metric to Fefferman-Graham like coordinates; the relevant details of this transformation can be found in Appendix C. Near the boundary, the metric can be written as

$$ds^2 = \left(\frac{L}{\tilde{z}}\right)^{\frac{2}{1-2\delta}} \left\{ \left[ -1 + \left(\frac{3-4\delta}{4-4\delta}\right) \left(\frac{\tilde{z}}{z_h}\right)^{2l} \right] dt^2 + \left[ 1 + \left(\frac{1}{4-4\delta}\right) \left(\frac{\tilde{z}}{z_h}\right)^{2l} \right] d\mathbf{x}^2 + d\tilde{z}^2 \right\} + \mathcal{O}\left(\frac{\tilde{z}^{4l}}{z_h^{4l}}\right). \quad (66)$$

One must now solve the equation of motion using this metric. Since we are working in the regime of  $Q \rightarrow \infty$ , we will denote the relevant fluctuation as  $H^\infty$ . Solving the equations of motion in this regime is rather technical, but details can be found in Appendix D. The near boundary solution is

$$H^\infty(\tilde{z}) \approx C_3 \left(\frac{\tilde{z}}{z_h}\right)^l \times \left[ K_l(Qz) - \left(\frac{\tilde{z}}{z_h}\right)^l \left(\frac{2}{Qz_h}\right)^l \frac{3\Gamma(l)}{4(5-4\delta)} + \mathcal{O}\left(\frac{\tilde{z}^{2l}}{z_h^{2l}}\right) \right] \quad (67)$$

Here  $C_3$  is a normalization constant, and  $K_l$  is a Bessel function. The reader is referred to Appendix D for more details.

To compute the correlator, we use the results for the on-shell action given previously (30), the Son and Starinets prescription gives

$$G_R(w = i\infty) = Ts \lim_{\epsilon \rightarrow 0} \frac{f(\epsilon)}{f'(\epsilon)} \frac{H_1^{\infty'}(Q, \epsilon)}{H_0^\infty(Q, \epsilon)} + G_R^{\text{CT}}(w = i\infty). \quad (68)$$

Here, the second term with the superscript  $CT$  denotes the contact terms which arise from the part of the on-shell action which contains no derivatives of  $H$ . We will have more to say about this term in a moment. Our solution (67) for  $H^\infty$  contains two terms; the first term is the result if one takes the horizon  $z_h \rightarrow \infty$ ; in other words, it is the zero temperature result. Upon subtracting the zero temperature piece, we have

$$\Delta G_R(w = i\infty) = Ts \lim_{\epsilon \rightarrow 0} \frac{f(\epsilon)}{f'(\epsilon)} \frac{H_1^{\infty'}(Q, \epsilon)}{H_0^\infty(Q, \epsilon)} + \Delta G_R^{\text{CT}}(w = i\infty). \quad (69)$$

Noting that

$$H_1^\infty = -\xi \left(\frac{\tilde{z}}{z_h}\right)^{2l}, \quad (70)$$

with  $\xi$  a constant, and

$$f(\tilde{z}) = 1 - \frac{\tilde{z}^{2l}}{z_h^{2l}} + \mathcal{O}\left(\frac{\tilde{z}^{4l}}{z_h^{4l}}\right) \quad (71)$$

we have,

$$\Delta G_R(w = i\infty) = Ts \frac{\xi}{H_0^\infty(Q, \epsilon)} + \Delta G_R^{\text{CT}}(w = i\infty). \quad (72)$$

Finally, with the fact that

$$H_\infty^0(Q, \epsilon) = \frac{1}{2} \left(\frac{2}{Qz_h}\right)^l \Gamma(l) + \mathcal{O}(\epsilon), \quad (73)$$

we come to the result

$$\Delta G_R(w = i\infty) = Ts \left[ \frac{3}{2(5-4\delta)} \right] + \Delta G_R^{\text{CT}}(w = i\infty). \quad (74)$$

Ostensibly, we are not finished as the left side of the sum rule also contains a term  $\Delta G_R(w = 0)$ , and we also have to deal with the contact terms arising from the part of the action containing no derivatives of  $H$ . There is also the issue of counter terms which could be added to regularize the action. The claim of [10] is that the contact terms above precisely cancel the contribution at  $w = 0$ , and that all counter terms will cancel if the zero temperature subtraction is done properly. We will not prove this claim here, but the agreement between our derived formula and our numerical results is an empirical justification for this claim. We believe that showing this explicitly should be mostly straightforward given our solutions and our expansions for the on-shell action; however the zero temperature subtraction can sometimes be a subtle issue (see for example, [38]). The final result of our analysis is

$$\Delta G_R(w = i\infty) - \Delta G_R(w = 0) = Ts \left[ \frac{3}{2(5-4\delta)} \right]. \quad (75)$$

Using now our improved formula we compare the left and right sides of the sum rule in Table II. As one can see, the agreement is much better, and the numerical results agree with the analytic ones to at least three significant figures. The fact that the error increases with  $\delta$  is not surprising, since the numerics become more challenging as  $\delta$  increases. This is because the power divergences of  $\rho$  and  $\rho_{T=0}$  are worse, and hence greater numerical precision is required. Furthermore, as  $\delta$  increases, the oscillations take longer to die out, and hence one must compute  $\Delta\rho$  to a larger value of  $\mathbf{w}$  in order to achieve the same accuracy.

The largest source of error in these computations is due to the fact that we only compute  $\rho$  up to a value  $\mathbf{w}_{\text{max}}$ , and our numerical integration stops at this value. As



$\delta$	$(\text{New Sum rule LHS}) \times \frac{1}{Ts}$	$(\text{Sum rule RHS}) \times \frac{1}{Ts}$
0.0	$\frac{3}{10} \approx 0.300000\dots$	0.3000(04)...
0.1	$\frac{15}{46} \approx 0.326087\dots$	0.3260(86)...
0.2	$\frac{5}{14} \approx 0.357143\dots$	0.3571(28)...
0.3	$\frac{15}{38} \approx 0.394737\dots$	0.395(036)...

TABLE II. Comparison of the left and right sides of the improved sum rule (77) which is now corrected to include the non-conformal effects of the theory. The left side is computed from (75), while the right side is computed by the numerical integration over the spectral density. The parentheses denote an estimate of numerical error which is explained more completely in the text.

shown in [10], the zero temperature subtracted spectral density is an oscillating function which dies out nearly exponentially at large  $\mathfrak{w}$ . We are neglecting the integral of this function from  $\mathfrak{w} = \mathfrak{w}_{\text{max}}$  to  $\mathfrak{w} = \infty$ . One can estimate the contribution of this “tail” by fitting the last few oscillations near  $\mathfrak{w}_{\text{max}}$  to a damped sine curve<sup>5</sup> of the form

$$\frac{\Delta\rho^{\text{shear}}(\mathfrak{w} \gg 1)}{\mathfrak{w}} \approx ae^{-b\mathfrak{w}} \sin(c\mathfrak{w} + d). \quad (76)$$

Once the parameters  $a, b, c$  and  $d$  are found from the fit, one can then integrate this function from  $\mathfrak{w}_{\text{max}}$  to  $\infty$  to estimate its contribution. We find that the order of magnitude of this tail contribution is  $\sim 10^{-5}$  for  $\delta = 0, 0.1$ , and  $0.2$ . The contribution of the tail appears to increase with  $\delta$ , and is roughly  $\sim 10^{-4}$  for  $\delta = 0.3$ . This is the significance of the parenthesis in the table of our numerical results; it is expected that the contribution of the tail will affect the numbers inside the parenthesis. However, it is quite clear that our analytic results agree with the numerical ones within our estimated numerical error.

The sum rule for the particular non-conformal theory in which we are working is

$$Ts \left[ \frac{3}{2(5-4\delta)} \right] = \frac{2}{\pi} \int_0^\infty \frac{dw}{w} \Delta\rho^{\text{shear}}(w). \quad (77)$$

It is desirable to write the left side in terms of thermodynamic observables, in hopes that our sum rule could be applicable to other theories beyond those considered here. Unfortunately, there is not a unique way of doing that in our case, as one could write either

$$\delta = \frac{\varepsilon - 3P}{2\varepsilon} \quad \text{or} \quad \delta = \frac{1}{2} (1 - 3v_s^2). \quad (78)$$

<sup>5</sup> This functional form was chosen for the purpose of error estimation only, and appears to fit the numerics quite well. In fact, the form we have chosen matches an analytical calculation of a current-current correlation function in [18]. However, we stress that the exact functional form of  $\Delta\rho^{\text{shear}}$  is not known analytically at large  $\mathfrak{w}$ , though it should be possible to calculate it using methods similar to those given in this paper.

This is a consequence of the fact that the speed of sound is independent of temperature in this model. One could perhaps gain more information by investigating this sum rule in a gravitational dual theory where the speed of sound depends on temperature (e.g. [22, 23, 38, 39])

We elect to write the left side in terms of  $\varepsilon$  and  $P$  only. Then, the sum rule can be written

$$\frac{\varepsilon(\varepsilon + P)}{2(\varepsilon + 2P)} = \frac{2}{\pi} \int_0^\infty \frac{dw}{w} \Delta\rho^{\text{shear}}(w). \quad (79)$$

This sum rule reduces to (1) in the case of  $\varepsilon = 3P$ . In other words, we have found that the function  $F(\varepsilon, P, v_s)$  defined in (4) can be written

$$F(\varepsilon, P, v_s) = \frac{\varepsilon(\varepsilon - 3P)}{10(\varepsilon + 2P)}. \quad (80)$$

## V. CONCLUSION

In this work, we have examined a sum rule involving a particular two point function of the energy-momentum tensor. A version of this sum rule was derived in [10], but we have explicitly shown that the sum rule given there is not valid for all Einstein gravity dual theories. We have provided a non-conformal generalization of this sum rule; the main result of this work is (79). We have numerically verified that the left side of our improved sum rule equals the right side with an accuracy greater than 0.1%. Whether our result is applicable to other non-conformal gravity duals, or even to non-conformal field theories should certainly be tested. To this effect, we have examined the sum rule in Yang-Mills theory in Appendix A. In addition, it is also important to numerically verify other sum rules given in [10]. These investigations are currently underway.

Finally, in the course of our investigation of the sum rule, we computed the spectral density in the tensor channel at various values of our non-conformal deformation parameter  $\delta$ . The behavior of the spectral density and the associated Euclidean correlation functions exhibit interesting qualitative behavior as a function of  $\delta$  as shown in Figs. 1 - 3. While the qualitative features of the spectral density and Euclidean correlation functions were found in the conformal case of  $\mathcal{N} = 4$  SYM theory [33, 34], to the best of our knowledge the change in these functions as a result of non-conformality has not been examined before. In particular, our results seem to suggest that the bulk viscosity (which is proportional to  $\delta$ ) may have strong effects on the shape and magnitude of the *tensor* Euclidean correlators measured on the lattice. This is noteworthy because the bulk viscosity is usually found from the Kubo relation which involves a different correlation function. It remains to be seen whether these qualitative features will persist in more detailed gravitational dual models which attempt to capture more features of the quark-gluon plasma.

## ACKNOWLEDGMENTS

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## Appendix A: Sum rule in Yang-Mills theory

We will use this section to make some comments on the shear sum rule in (weakly coupled) Yang-Mills theory. A tentative form for this sum rule was written in [10]; but here we will perform the calculation in a slightly different way.

The left side of the sum rule contains correlation functions at large frequency. Such quantities can be calculated using the operator product expansion (OPE). In the limit of  $w \rightarrow \infty$ , one only needs to know the OPE to leading order. In [40], the leading order OPE was calculated for the correlator of interest, with the result (in Euclidean signature)

$$G^{\text{shear}}(q \rightarrow \infty) = \frac{2}{3q^2} (q_4^2 - \vec{q}^2) \langle T^{44} \rangle + \frac{1}{6} \langle F^2 \rangle \quad (\text{A1})$$

Here,  $F_{\mu\nu}^a$  is the gluon field strength tensor, and  $F^2 \equiv F_{\mu\nu}^a F_{\mu\nu}^a$ . The stress-energy tensor is denoted by  $T^{\mu\nu}$ ,

$$T_{\mu\nu} = F_{\mu\lambda}^a F_{\nu}^{a\lambda} - \frac{1}{4} g_{\mu\nu} F^2, \quad (\text{A2})$$

and the index “4” pertains to  $q_4 = iw$ . In the limit of zero spatial momentum, and returning to Minkowski signature, we find

$$G^{\text{shear}}(w \rightarrow i\infty, \vec{q} \rightarrow 0) = -\frac{2}{3} \langle T^{00} \rangle + \frac{1}{6} \langle F^2 \rangle. \quad (\text{A3})$$

All that remains is to express these quantities in terms of thermodynamic ones; the arguments here follow [41]. With the assumption of a perfect fluid, and with the use of the trace anomaly we can write

$$\langle T^{00} \rangle = \varepsilon, \quad (\text{A4})$$

$$\langle F^2 \rangle = -\frac{2g}{\beta(g)} \langle T_{\mu}^{\mu} \rangle = \frac{2g}{\beta(g)} (\varepsilon - 3P), \quad (\text{A5})$$

where the scale dependence on the left hand side of (A5) is transformed to the scale dependence of the coupling constant. For notational convenience, we define

$$G^{\text{shear}}(w \rightarrow i\infty, \vec{q} \rightarrow 0) \equiv G_{\infty}^{\text{shear}}. \quad (\text{A6})$$

We find

$$G_{\infty}^{\text{shear}} = -\frac{2}{3} \varepsilon + \frac{g}{3\beta(g)} (\varepsilon - 3P). \quad (\text{A7})$$

To leading order, this becomes:

$$-G_{\infty}^{\text{shear}} = \frac{2}{3} \varepsilon + \frac{4\pi}{11N_c \alpha_s} (\varepsilon - 3P). \quad (\text{A8})$$

This is the quantity that will enter the left side of the sum rule for pure gluodynamics, the low energy term  $\Delta G_R(w=0)$  vanishes as shown in Appendix A of [10]. Note that the second term in (A8) which is proportional to  $\alpha_s^{-1}$  was not given in [10], though it appears to have been independently noticed in a recent paper [13]. This additional term originates from the  $\langle F^2 \rangle$  term in A1. In [40], this term is argued to be a contact term due to the fact that it renders the correlation function non-transverse. If this is the case, this additional term should not be present in the sum rule, since all contact terms will cancel out due to the subtraction  $\Delta G_R(w=i\infty) - \Delta G_R(w=0)$ . Despite the claim made in [40], we are unaware of any explicit calculation which shows that  $\Delta G_R^{\text{shear}}(w=0) \sim \varepsilon - 3P$ . For this reason, we currently choose to include the additional term in the sum rule, though it is clear that more work should be done to address this issue in the future.

In perturbation theory,  $\varepsilon - 3P \sim \mathcal{O}(\alpha_s^2)$ , so in total, the second term on the right side of (A8) is  $\mathcal{O}(\alpha_s)$ . Let us make a few comments on what may happen to this expression as we go beyond leading order in perturbation theory. We expect that the Wilson coefficients in (A1) will, in general, contain corrections of  $\mathcal{O}(\alpha_s)$ . In other words, we could expect that (A1) becomes

$$G^{\text{shear}}(q \rightarrow \infty) = \frac{2}{3q^2} (1 + a_1 \alpha_s) (q_4^2 - \vec{q}^2) \langle T^{44} \rangle + \frac{1}{6} (1 + a_2 \alpha_s) \langle F^2 \rangle \quad (\text{A9})$$

with constants  $a_1$  and  $a_2$  to be determined from a one loop calculation. Following through the previous arguments, we would arrive at the expression

$$-G_{\infty}^{\text{shear}} = \frac{2}{3} (1 + a_1 \alpha_s) \varepsilon + \frac{4\pi}{11N_c \alpha_s} (\varepsilon - 3P) + \mathcal{O}(\alpha_s^2). \quad (\text{A10})$$

or, equivalently,

$$-G_{\infty}^{\text{shear}} = \frac{1}{2} (1 + a_1 \alpha_s) (\varepsilon + P) + \frac{4\pi}{11N_c \alpha_s} (\varepsilon - 3P) + \mathcal{O}(\alpha_s^2). \quad (\text{A11})$$

where again we have used the fact that  $\varepsilon - 3P$  is  $\mathcal{O}(\alpha_s^2)$ .

We find it interesting that the left side of the sum rule in our strongly coupled model has a different dependence on  $\varepsilon$  and  $P$  than the left side of the sum rule in weakly coupled Yang-Mills theory. It is not clear whether this has any implications for the possibility of a gravity dual of *weakly coupled* Yang-Mills theory.

## Appendix B: Background Equations

For the black brane type metric generated by a single scalar field, with a  $z$  denoting the radial coordinate, the

background equations of motion can be written

$$\partial_z [\sqrt{-g} g^{zz} \mathcal{D}_L[f]] = 0 \quad (\text{B1})$$

$$\frac{3}{2\sqrt{-g}} \partial_z [\sqrt{-g} g^{zz} \mathcal{D}_L[g_{xx}]] = -V(\phi) \Big|_{\phi=\phi_0(z)} \quad (\text{B2})$$

$$3\mathcal{D}_L[g_{xx}] \mathcal{D}_L \left[ \frac{\sqrt{g_{zz}f}}{\mathcal{D}_L[g_{xx}]} \right] = \phi'_0(z)^2 \quad (\text{B3})$$

$$\frac{1}{\sqrt{-g}} \partial_z [\sqrt{-g} g^{zz} \phi'_0(z)] = \frac{\partial V}{\partial \phi} \Big|_{\phi=\phi_0(z)}, \quad (\text{B4})$$

where  $f$  and  $\mathcal{D}_L$  are defined as in the text (9) and (28) respectively.

### Appendix C: Transformation to Fefferman-Graham like coordinates

In this section, we detail the transformation of the Chamblin-Reall metric to a Fefferman-Graham like coordinate system. Fefferman-Graham coordinates are useful in many respects for asymptotically anti de-Sitter metrics; these coordinates are defined so that the metric takes the form

$$ds^2 = \frac{L^2 (\tilde{g}_{\mu\nu} dx^\mu dx^\nu + d\tilde{z}^2)}{\tilde{z}^2}, \quad (\text{C1})$$

where the indices  $\mu$  and  $\nu$  run over the four coordinates  $t, x^i$ .

Note that the Chamblin-Reall metric is not asymptotically anti de-Sitter (except for the case of  $\delta = 0$ ), hence we will make a slight generalization of the Fefferman-Graham coordinates, where the metric is written

$$ds^2 = \frac{L^n (\tilde{g}_{\mu\nu} dx^\mu dx^\nu + d\tilde{z}^2)}{\tilde{z}^n}. \quad (\text{C2})$$

It is not always possible to solve for  $\tilde{g}_{\mu\nu}$  analytically, but one can find its behavior near the boundary as an expansion in the radial coordinate. To transform our metric to this form, we begin with the Chamblin-Reall metric written in the coordinate system 11, and apply the coordinate transformation [42],

$$\left(\frac{L}{z}\right)^n \frac{dz^2}{f(z)} = \left(\frac{L}{\tilde{z}}\right)^n d\tilde{z}^2. \quad (\text{C3})$$

Enforcing the fact that near the boundary,  $z = \tilde{z}$ , we can integrate this equation to find

$$\int_\epsilon^z \frac{dz}{z^{n/2} \sqrt{f(z)}} = \int_\epsilon^{\tilde{z}} \frac{d\tilde{z}}{\tilde{z}^{n/2}}. \quad (\text{C4})$$

For the time being, let us assume that  $n \neq 2$ , (though we will be able to relax this assumption later on). Then, performing the integral we find

$$\tilde{z}^{1-\frac{n}{2}} = \left(1 - \frac{n}{2}\right) \int_\epsilon^z \frac{dz}{z^{n/2} \sqrt{f(z)}} + \epsilon^{1-\frac{n}{2}} \quad (\text{C5})$$

$$= z^{1-\frac{n}{2}} + \left(1 - \frac{n}{2}\right) \int_0^z \frac{1}{z^{n/2}} \left[ \frac{1}{\sqrt{f(z)}} - 1 \right] dz. \quad (\text{C6})$$

We have taken the limit  $\epsilon \rightarrow 0$  freely since the integral converges. Near the boundary, one can expand the integrand to find<sup>6</sup>

$$\tilde{z}^{1-\frac{n}{2}} \approx z^{1-\frac{n}{2}} \left\{ 1 + \left(1 - \frac{n}{2}\right) \frac{1}{4l-n+2} \left(\frac{z}{z_h}\right)^{2l} + \mathcal{O}\left(\frac{z^{2l}}{z_h^{2l}}\right) \right\}. \quad (\text{C7})$$

(Note that  $f(z) = 1 - (z/z_h)^{2l}$ .) Then, to this order we have,

$$\tilde{z} \approx z \left\{ 1 + \frac{1}{4l-n+2} \left(\frac{z}{z_h}\right)^{2l} + \mathcal{O}\left(\frac{z^{4l}}{z_h^{4l}}\right) \right\}, \quad (\text{C8})$$

$$z \approx \tilde{z} \left\{ 1 - \frac{1}{4l-n+2} \left(\frac{\tilde{z}}{z_h}\right)^{2l} + \mathcal{O}\left(\frac{\tilde{z}^{4l}}{z_h^{4l}}\right) \right\}. \quad (\text{C9})$$

In terms of  $\delta$ , this is,

$$z \approx \tilde{z} \left\{ 1 - \frac{1-2\delta}{8-8\delta} \left(\frac{\tilde{z}}{z_h}\right)^{2l} + \mathcal{O}\left(\frac{\tilde{z}^{4l}}{z_h^{4l}}\right) \right\}. \quad (\text{C10})$$

Strictly speaking, we derived this result assuming  $n \neq 2$ , or equivalently  $\delta \neq 0$ . However, if one redoes the analysis for  $\delta = 0$ , one finds precisely the same result as that given by (C10), thus (C10) is valid for all values of  $\delta$  in the physical regime  $0 \leq \delta < \frac{1}{2}$ . Applying this coordinate transform, we find the Fefferman-Graham like representation of the Chamblin-Reall metric which is given in the text (66) to order  $\mathcal{O}(z^{2l}/z_h^{2l})$ . Since we are only interested in the near-boundary dynamics, we need not worry about higher order terms.

### Appendix D: Calculation of $H^\infty$

In this section we present some of the technical details involved in solving the equations of motion in the large  $w$  regime. Throughout this section we work in the Fefferman-Graham coordinate system, which is explained in detail in Appendix C. Perhaps the easiest way to get the relevant equation is to introduce a scaling parameter  $\lambda$ , by replacing  $\left(\frac{\tilde{z}}{z_h}\right)^{2l} \rightarrow \lambda \left(\frac{\tilde{z}}{z_h}\right)^{2l}$ , making the ansatz

$$H^\infty(\tilde{z}) = H_0^\infty(\tilde{z}) + \lambda H_1^\infty(\tilde{z}). \quad (\text{D1})$$

and expanding the equation of motion (21) in  $\lambda$ . The lowest order equation gives the  $T = 0$  equation for  $H_0$  (with  $w^2$  replaced by  $-Q^2$ ).

$$(H_0^\infty)'' - \frac{3}{\tilde{z}(1-2\delta)} (H_0^\infty)' - Q^2 H_0^\infty = 0. \quad (\text{D2})$$

<sup>6</sup> We assumed  $2l - \frac{n}{2} \neq -1$ . This is valid provided  $\delta \neq 5/2$  which is outside the physical regime of interest

The first order equation is

$$(H_1^\infty)'' - \frac{3}{\tilde{z}(1-2\delta)}(H_1^\infty)' - Q^2 H_1^\infty = \left(\frac{\tilde{z}}{z_h}\right)^{2l} \frac{1}{1-\delta} \left[ \frac{3-4\delta}{4} Q^2 H_0^\infty - \delta l \frac{(H_0^\infty)'}{\tilde{z}} \right] \quad (\text{D3})$$

The solution for  $H_0$  is

$$H_0^\infty(Q, \tilde{u}) = C_3 \tilde{u}^l K_l(Q \tilde{u} z_h) + C_4 \tilde{u}^l I_l(Q \tilde{u} z_h), \quad (\text{D4})$$

where we have defined  $\tilde{u} \equiv \tilde{z}/z_h$  for convenience.  $I_l$  and  $K_l$  are the modified Bessel functions of the first and second kind respectively. Regularity at  $\tilde{z} \rightarrow \infty$  [10, 17] requires  $C_4 = 0$ .

The homogeneous part of the first order equation (D4) is the same as the zero-temperature equation, hence this equation can be solved with the use of a Green's function. Defining the two homogeneous solutions as

$$y_1(\tilde{u}) = C_3 \tilde{u}^l K_l(Q \tilde{u} z_h) \quad (\text{D5})$$

$$y_2(\tilde{u}) = C_3 \tilde{u}^l I_l(Q \tilde{u} z_h), \quad (\text{D6})$$

The solution is

$$H_1^\infty(\tilde{u}) = -\frac{1}{C_3^2} \left\{ y_1(\tilde{u}) \int_0^{\tilde{u}} y_2(t) g(t) dt + y_2(\tilde{u}) \int_{\tilde{u}}^\infty y_1(t) g(t) dt \right\}, \quad (\text{D7})$$

where

$$g(\tilde{u}) = \frac{\tilde{u}}{1-\delta} \left[ \frac{3-4\delta}{4} (Q z_h)^2 H_0^\infty - \delta l \frac{(H_0^\infty)'}{\tilde{u}} \right] = \frac{\tilde{u}}{1-\delta} \left[ \frac{3-4\delta}{4} (Q z_h)^2 y_1(\tilde{u}) - \delta l \frac{y_1'(\tilde{u})}{\tilde{u}} \right]. \quad (\text{D8})$$

In order to compute the correlation function, we will only need the leading term at the boundary  $\tilde{u} = 0$ ,

$$H_1(\tilde{u} \rightarrow 0) = -\frac{1}{C_3^2} \left\{ y_2(u \rightarrow 0) \int_0^\infty y_1(t) g(t) dt \right\} \quad (\text{D9})$$

which can be written:

$$H_1(\tilde{u} \rightarrow 0) = \frac{1}{C_3(\delta-1)} \left\{ \left( \frac{\tilde{u}^2 Q z_h}{2} \right)^l \frac{1}{l\Gamma(l)} \times \int_0^\infty \left[ \frac{3-4\delta}{4} (Q z_h)^2 t y_1(t)^2 - \frac{\delta l}{2} [y_1(t)^2]' \right] dt \right\}. \quad (\text{D10})$$

The first term in the integrand requires the integral

$$\int_0^\infty dt K_l(t)^2 t^{2l+1} = \frac{\Gamma(1+l)\Gamma(1+2l)\sqrt{\pi}}{4\Gamma(\frac{3}{2}+l)}, \quad (\text{D11})$$

and the second term in the integrand is a total derivative which reduces to the boundary term  $\sim y_1(0)^2$ , (note that  $y_1(\tilde{u} \rightarrow \infty)$  approaches zero exponentially). In total, the result is

$$H_1^\infty(\tilde{z} \rightarrow 0) = \frac{C_3}{4(1-\delta)} \left( \frac{\tilde{z}^2}{2Q z_h^3} \right)^l \times \left[ \frac{3-4\delta}{4} \frac{\Gamma(1+2l)\sqrt{\pi}}{\Gamma(\frac{3}{2}+l)} + \frac{\delta l^l}{2} \Gamma(l) \right]. \quad (\text{D12})$$

The identity

$$\Gamma(l+1)\Gamma\left(l+\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \frac{1}{4^l} \Gamma(2l+2) \quad (\text{D13})$$

can be used to simplify the result to:

$$H_1^\infty(\tilde{z} \rightarrow 0) = -C_3 \left( \frac{2\tilde{z}^2}{Q z_h^3} \right)^l \Gamma(l) \left( \frac{3}{4(5-4\delta)} \right). \quad (\text{D14})$$

In summary, the solution for  $H$  for large values of  $Q$ ,  $H(w = i\infty) \equiv H_\infty$  can be written

$$H^\infty(\tilde{z}) \approx C_3 \left( \frac{\tilde{z}}{z_h} \right)^l \times \left[ K_l(Qz) - \left( \frac{\tilde{z}}{z_h} \right)^l \left( \frac{2}{Q z_h} \right)^l \frac{3\Gamma(l)}{4(5-4\delta)} + \mathcal{O}\left(\frac{\tilde{z}^{2l}}{z_h^{2l}}\right) \right] \quad (\text{D15})$$

This is the solution which is quoted in the text.

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